

A nonlinear failed cartel: existence, optimality and computation

Companion to A Failed Cartel: When Collusive Firms Become Competitors

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Main point. Identical firms can produce different amounts because they play different strategic roles. Firms 1 and 2 choose first, anticipating how Firm 3 will respond; Firm 3 observes their choices before deciding its own output. When the first two firms cooperate, they restrain production to protect their joint profit, and the outsider produces more than either member. When they compete instead, each expands production and the follower produces less. The technology and order of decisions stay the same: changing cooperation alone reverses the output ranking.

The nonlinear demand makes this intuition something to prove, rather than infer from first-order conditions. Price diverges as total output approaches zero, quantities have no upper bound, and the follower may optimally shut down. We establish that best responses exist and verify global optimality, including boundary choices, in both games. Separately, for the coalition problem, we construct a contraction with explicit numerical error bounds. This distinguishes an economically valid equilibrium from a stationary solution and a convergent computation.

1. Primitives and the issues that require proof

Selling more lowers the market price, while every extra unit costs the same to produce. We must check that a finite best choice exists.

Let $x, y, z \geq 0$, $Q = x + y + z$, and take the article's Section 5 isoelastic specification with marginal cost normalized to one:

$$P(Q) = Q^{-1/2} \quad (Q > 0), \quad C(q) = q, \quad \pi_i(q_i, Q) = q_i(Q^{-1/2} - 1). \quad (1)$$

At $Q = 0$, define every firm's profit to be zero. This is a continuous extension because $0 \leq q_i/\sqrt{Q} \leq \sqrt{Q}$. No price at zero is needed. The coalition, when present, chooses its total $s = x + y$ at stage 1 and imposes $x = y = s/2$; at stage 2 the outsider observes s and chooses z . Linear costs make the allocation of a given s indeterminate without this equal-sharing rule. Membership and timing are exogenous.

Continuity on unbounded strategy sets does not ensure attainment. With zero marginal cost, the follower's profit $z/\sqrt{s+z}$ would grow without bound as $z \rightarrow \infty$, leaving no finite best response. Positive cost is therefore essential here.

2. A globally optimal follower after every history

The outsider balances extra sales against their effect on price. If the leaders already supply enough, producing nothing is its best option.

For each $s \geq 0$, the follower maximizes $V(z; s) = z/\sqrt{s+z} - z$ on $z \geq 0$, with the preceding extension at $(s, z) = (0, 0)$. Since

$$V(z; s) \leq \sqrt{z} - z \rightarrow -\infty,$$

a maximum is attained. For $s + z > 0$,

$$V_z = (s + z)^{-1/2} - \frac{z}{2}(s + z)^{-3/2} - 1, \quad V_{zz} = -\frac{s + z/4}{(s + z)^{5/2}} < 0.$$

For $s > 0$, the derivative at zero is $s^{-1/2} - 1$. Strict concavity therefore gives a unique best response $b(s)$, with $b(s) = 0$ exactly when $s \geq 1$. At $s = 0$, maximizing $\sqrt{z} - z$ gives $b(0) = 1/4$.

On the active branch $0 \leq s < 1$, set $u = \sqrt{s + b(s)}$. The stationary equation then gives

$$s = S(u) := u^2(2u - 1), \quad b(s) = Z(u) := 2u^2(1 - u), \quad \frac{1}{2} \leq u < 1. \quad (2)$$

Since $S'(u) = 2u(3u - 1) > 0$, S maps $[1/2, 1)$ bijectively onto $[0, 1)$, defining a unique response that joins continuously to shutdown at $s = 1$.

3. Why an interior stationary calculation is insufficient

A formula for positive production cannot tell the whole story: it must also identify when the firm should stop producing. The full continuation rule is the implicit branch (2) for $s < 1$ and zero for $s \geq 1$. On the active branch,

$$b'(s) = \frac{2 - 3u}{3u - 1}.$$

The follower initially expands, then contracts; at shutdown the derivative jumps from $-1/2$ to zero. Thus the response is not globally differentiable. We use interior derivatives only for $s < 1$ and check shutdown separately. Optimality after every history, proved above, supports the subgame-perfect interpretation.

Global equilibrium under cooperation and competition

4. The surviving coalition: existence and global optimality

The coalition balances its combined sales against the price reduction they cause, allowing for the outsider's response. It must beat every alternative, including producing nothing or enough to drive the outsider out.

Anticipating the verified follower rule, the coalition maximizes

$$J(s) = s((s + b(s))^{-1/2} - 1), \quad s \geq 0.$$

For $s \geq 1$, $b(s) = 0$ and $J(s) = \sqrt{s} - s \leq 0$, tending to $-\infty$. At $s = 0$, $J(0) = 0$. On $0 < s < 1$, use the increasing coordinate u in (2):

$$J(S(u)) = W(u) := u(1 - u)(2u - 1), \quad \frac{1}{2} \leq u \leq 1. \quad (3)$$

This continuous polynomial is positive inside $(1/2, 1)$ and zero at its endpoints. Hence its maximum is attained in the interior and dominates the entire shutdown branch. Moreover,

$$W'(u) = -6u^2 + 6u - 1, \quad W''(u) = 6 - 12u < 0 \quad (1/2 < u < 1).$$

There is exactly one feasible stationary point,

$$u_{\text{FC}} = \frac{3 + \sqrt{3}}{6}, \quad (x^{\text{FC}}, y^{\text{FC}}, z^{\text{FC}}) = H(u_{\text{FC}}), \quad H(u) := \left(\frac{u^2(2u - 1)}{2}, \frac{u^2(2u - 1)}{2}, 2u^2(1 - u) \right). \quad (4)$$

It is the unique global coalition aggregate optimum. The other algebraic root $(3 - \sqrt{3})/6 < 1/2$ gives $S(u) < 0$ and is infeasible. Thus even the stationary polynomial must be interpreted on its valid branch. With equal sharing and the complete follower rule, (4) is a subgame-perfect outcome. Without equal sharing, all allocations with the same optimal s give the same joint profit; uniqueness does not extend to the individual coalition quantities.

5. Same timing, but two competing leaders

Each leader now values its own extra sales without counting the loss imposed on the other leader. This encourages greater output even though the follower still moves last.

Now Firms 1 and 2 choose $x, y \geq 0$ independently at stage 1; the follower still uses exactly the rule in Section 2. Fix $y < 1$. On the active branch the leader's reduced profit, expressed in the increasing coordinate u , is

$$F_y(u) = [S(u) - y](u^{-1} - 1) = W(u) - y(u^{-1} - 1), \quad S(u) \geq y, \quad u < 1.$$

Its derivatives satisfy

$$F'_y(u) = -6u^2 + 6u - 1 + \frac{y}{u^2}, \quad F''_y(u) = 6 - 12u - \frac{2y}{u^3} < 0 \quad (u > 1/2).$$

At the lower endpoint $S(u) = y$, the right derivative is $S'(u)(u^{-1} - 1) > 0$; at $u = 1$, $F'_y(1) = y - 1 < 0$. Hence there is a unique interior global maximizer on this branch, with positive profit. Any choice with $x + y \geq 1$ gives nonpositive profit, so it cannot improve on this maximum. If $y \geq 1$, every $x > 0$ has negative profit and the unique best response is $x = 0$. This verifies global best responses on the entire unbounded strategy set, without assuming concavity in x .

Neither leader can be inactive at equilibrium. If both are zero, entry pays. If only one is active, its problem is the coalition aggregate problem above, which gives $s = S(u_{\text{FC}}) < 1$; the inactive leader can then enter profitably. Both leaders must therefore have positive output with $s < 1$.

Write $h(s) = 1/u(s) - 1$, so $h'(s) = -1/[2u^3(3u - 1)] < 0$. The two first-order conditions are $h + xh' = 0$ and $h + yh' = 0$, implying $x = y = s/2$. Substitution gives

$$12u^2 - 14u + 3 = 0, \quad u_{\text{CS}} = \frac{7 + \sqrt{13}}{12}, \quad (x^{\text{CS}}, y^{\text{CS}}, z^{\text{CS}}) = H(u_{\text{CS}}). \quad (5)$$

The other root lies below $1/2$. The global best-response proof establishes that (5) is the unique leader Nash outcome, and the complete follower strategy makes it subgame perfect. Here $Q = u^2$ and $P = 1/u$ in both regimes. The formulas imply $u_{\text{FC}} < 5/6 < u_{\text{CS}}$: the outsider produces more than either coalition member, but less than either competing leader, since $Z(u) - S(u)/2 = u^2(5/2 - 3u)$.

A certified nonlinear solver and the scope of the result

6. Invariance, contraction and an error certificate

A useful calculation should do more than settle down: it should guarantee convergence and tell us how far the current answer may be from the solution.

For the partial cartel, rearrange $6u^2 - 6u + 1 = 0$ as

$$T(u) = 1 - \frac{1}{6u}, \quad D = \left[\frac{3}{4}, \frac{5}{6} \right].$$

Direct endpoint evaluation and differentiation give

$$T(D) = \left[\frac{7}{9}, \frac{4}{5} \right] \subset D, \quad 0 < T'(u) = \frac{1}{6u^2} \leq \frac{8}{27} =: \kappa < 1.$$

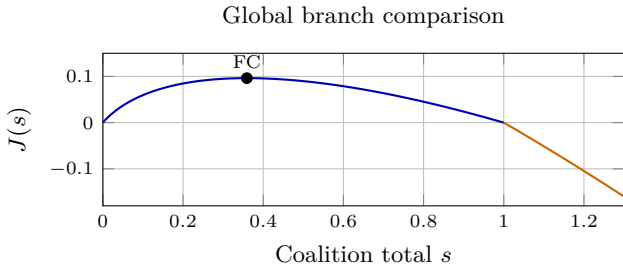
The interval is complete. Banach's theorem therefore yields a unique fixed point in D and convergence of $u_{m+1} = T(u_m)$ from every $u_0 \in D$. The fixed point is u_{FC} from (4). For $m \geq 1$,

$$|u_m - u_{\text{FC}}| \leq E_m := \frac{\kappa}{1 - \kappa} |u_m - u_{m-1}| = \frac{8}{19} |u_m - u_{m-1}|, \quad \|H(u_m) - H(u_{\text{FC}})\|_1 \leq \frac{10}{3} E_m. \quad (6)$$

Indeed, on D the first two components of H increase and the third decreases, so $\sum_i |H'_i(u)| = 6u(2u - 1) \leq 10/3$. Every iterate gives positive quantities and an optimal follower continuation. Starting at $u_0 = 3/4$ gives the following reproducible values:

m	u_m	E_m	Quantity-error bound $(10/3)E_m$
1	0.777777777778	1.170×10^{-2}	3.899×10^{-2}
4	0.788461538462	2.454×10^{-4}	8.179×10^{-4}
8	0.788674033149	1.268×10^{-6}	4.224×10^{-6}
19	0.788675134594	6.477×10^{-13}	2.159×10^{-12}

The bounds apply to the exact iterates: displayed bounds are rounded upward, and iterates to the nearest shown digit. The interval D is a numerical certification region, not a restriction on economic choices. Contraction proves convergence and uniqueness *within* D ; Section 4 separately proves global optimality over all $s \geq 0$ and excludes other economically relevant solutions.



Same primitives and timing

	Coalition	Competing leaders
$x = y$	0.179558	0.299781
z	0.262892	0.181533
Q	0.622008	0.781095
P	1.267949	1.131483
$\pi_1 = \pi_2$	0.048113	0.039416
π_3	0.070442	0.023868

Profits are $q_i(P - 1)$; entries are rounded.

Figure 1. The positive interior coalition maximum dominates both zero output and the entire shutdown branch (orange, shown only to $s = 1.3$). The table compares the two verified global outcomes.

7. What this illustrates—and what it does not

Finding a number and showing that firms would choose it are separate tasks. This example connects the two, while keeping their justifications distinct.

Nonlinearity alone does not make global optimality fragile: the transformation makes this particular example tractable. The scalar solver is the article's Section 5 construction, justified by Appendix B's contraction theorem. It is *not* the three-dimensional Cournot–Stackelberg operator G_2 or a verification of Theorem 5 for that operator. No contraction for the competing-leader game is claimed; its equilibrium was proved directly. Numerical iteration is not an economic adjustment process, and neither comparison establishes endogenous cartel breakdown or membership stability.

Source and scope. Badev et al., *A Failed Cartel: When Collusive Firms Become Competitors*, *Games* **17** (2026), 56, Sections 2.5 and 5, Proposition 14 and Appendix B.3; doi:10.3390/g17050056. Sections 1–4 and 6 unpack that article's isoelastic example at $c = 1$; Section 5 adds a direct same-timing comparison. The supplied *Example-Set-Valued* is a presentation model from a different manuscript. No set-valued existence theorem, response plateau, or generalized-concavity claim is imported.