

Generic Uniqueness and Selection in a Sequential Partial-Cartel Market

An example with production adjustment costs and small marginal cost differences

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Main point. Small production adjustment costs can switch a coalition between distant, nearly tied plant allocations. Independently varying marginal charges generically give a unique optimum and stable approximation despite kinks and nonconcavity, even when adjustment costs are zero.

1. A regular market with a uniquely responding outsider

Firms 1 and 2 jointly choose plant outputs $x = (x_1, x_2) \in X = [0, 5]^2$. Firm 3 observes their plan and chooses $z \in [0, 2]$. With $s = x_1 + x_2$ and $Q = s + z$, let

$$P(Q) = 20 - Q + \frac{1}{5} \sin(3Q), \quad c_1(u) = 8u + \frac{1}{20}u^2, \quad c_2(u) = \frac{41}{5}u + \frac{1}{10}u^2, \quad c_3(z) = 8z + 4z^2.$$

Assumptions (A1)–(A3): capacities are compact; demand is continuous, positive and decreasing; costs are continuous, convex and nondecreasing. Plant 2 has higher marginal costs. On $[0, 12]$, $P \geq 7.8$, $-1.6 \leq P' \leq -0.4$, and $|P''| \leq 1.8$. The follower's marginal profit satisfies

$$g_s(z) = P(s + z) + zP'(s + z) - c'_3(z), \quad \partial_z g_s = 2P' + zP'' - 8 \leq -5.2.$$

For every $s \in [0, 10]$, $g_s(0) \geq 1.8$ and $g_s(2) \leq -6.6$. Thus **(A4)** holds: the outsider has one interior global best reply $r(s)$, smooth in s and computable by bisection.

2. Staffing plans and the plant production allocation

The coalition has two staffing plans. Plan A books labor for two-thirds of output at Plant 1 and adds a supervisor costing 1.05 per period. Plan B books two-thirds at Plant 2 under existing supervision, with no additional overhead. Outputs remain free; each unit reassigned from the booked split costs 6 in overtime or idle labor. The *staffing overhead and mismatch cost* is

$$C(x) = \min\{1.05 + 6|x_1 - \frac{2}{3}s|, 6|x_1 - \frac{1}{3}s|\}, \quad J(x) = sP(s + r(s)) - c_1(x_1) - c_2(x_2) - C(x).$$

At total output six, the booked allocations are (4, 2) and (2, 4). Their profits are about 30.5793 and 30.6293, while equal sharing gives only $J(3, 3) \approx 25.2793$. The midpoint inequality disproves concavity. Continuity and compactness ensure an optimum, while Figure 1 shows competing local peaks.

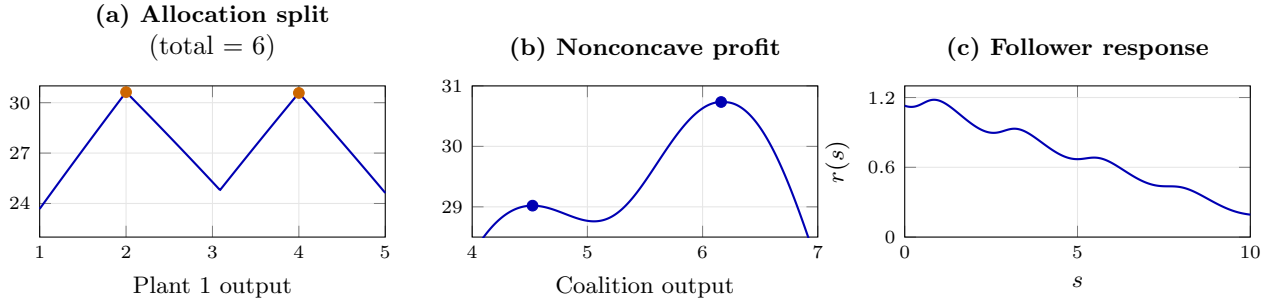


Figure 1: (a) Plant 1's share of six units; Plant 2 supplies the remainder. (b) Baseline profit maximized over the plant split at each total has competing local peaks. Both profit panels include staffing overhead and mismatch costs, and exclude production adjustment costs and marginal charges. (c) The outsider's unique response.

3. Economic costs that fit the local perturbation framework

Fix the inherited plan $b = (4, 2)$ before optimization. Revising material orders or machine settings incurs *production adjustment costs*; temporary financing or handling adds marginal charges:

$$A_{\kappa, \tau}(x) = \underbrace{\frac{\kappa}{2} \|x - b\|_2^2}_{\text{production adjustment cost}} + \underbrace{\tau_1 x_1 + \tau_2 x_2}_{\text{temporary marginal charges}}, \quad J_{\kappa, \tau} = J - A_{\kappa, \tau}, \quad \kappa, \tau_i \geq 0.$$

The parameters are fixed before optimization. C prices current staffing; the quadratic term prices revisions from b ; marginal charges alter output costs. On compact X , $q = C + A_{\kappa, \tau}$ is bounded and continuous, hence lower semicontinuous; $p = 0$ is upper semicontinuous. Convexity of q is unnecessary, and the follower response is unchanged.

4. A different allocation with little baseline-profit loss

With $\tau = (0.005, 0.005)$ and no adjustment costs, the numerical optimum uses staffing plan B and outputs $x \approx (2.0534, 4.1067)$. Adding $\kappa = 0.01$ selects plan A and $x \approx (4.1279, 2.0639)$, close to b . Relative to the baseline optimum, outputs move about 2.91 units, while baseline profit falls by only 0.002378. Raising Plant 1's charge to 0.04 selects B again.

Write $J^* = \max_X J$. Since $\max_X \|x - b\|_2^2 = 25$, comparison with a baseline maximizer gives, for any revised maximizer $x_{\kappa, \tau}$,

$$0 \leq A_{\kappa, \tau}(x) \leq M_{\kappa, \tau} := 12.5\kappa + 5(\tau_1 + \tau_2), \quad \boxed{0 \leq J^* - J(x_{\kappa, \tau}) \leq M_{\kappa, \tau}.$$

For $\kappa = 0.01$ and equal charges of 0.005, $M = 0.175$. This bounds baseline-profit loss; after-cost payoff also subtracts the new expenses. A δ -optimal revised plan has baseline loss at most $M + \delta$.

Locality. The bound $\|q - C\|_\infty = \|A\|_\infty \leq M$ gives uniform proximity of criteria as *both* κ and τ vanish, without imposing a search radius. Revised optimizers approach the baseline maximizing set; convergence to one plan requires uniqueness or further selection. Shrinking only τ with $\kappa > 0$ fixed leaves adjustment costs in the limit.

5. What the perturbation adds: uniqueness and stable approximation

For fixed b and $\kappa \geq 0$, *including* $\kappa = 0$, independently varying marginal charges give a unique optimum for *Lebesgue-almost every* τ in any arbitrarily small positive rectangle. To see why, define

$$V_\kappa(\tau) = \max_{x \in X} \{J(x) - \frac{\kappa}{2}\|x - b\|_2^2 - \tau \cdot x\}.$$

As a maximum of affine functions of τ , V_κ is convex; bounded X makes it Lipschitz, hence differentiable almost everywhere. Every optimizer x supplies a subgradient $-x$: for all σ , $V_\kappa(\sigma) \geq V_\kappa(\tau) - x \cdot (\sigma - \tau)$. At a differentiability point every optimizer must equal $-\nabla V_\kappa(\tau)$, proving uniqueness without concavity of J .

This finite-parameter argument complements the paper's function-space genericity theorem, which does not automatically apply to every parametric family. Neither adjustment costs alone nor unequal charges guarantee uniqueness at every prescribed pair. Equal-charge cases lie on a line and are not certified by the two-dimensional almost-everywhere result. The numerical comparisons above do not certify uniqueness.

The practical gain. At any fixed revised criterion with a unique optimum x^* , continuity and compactness force *every maximizing sequence* to converge to x^* , with follower output converging to $r(x_1^* + x_2^*)$. Optimizers of uniformly converging nearby criteria also converge to x^* . A brute-force grid search can approximate values; these properties explain when increasingly accurate computations identify one production plan. They give no contraction or rate; allocations can switch at ties. This is one planning decision with an inherited reference, not repeated adjustment dynamics.

6. What the perturbation approach adds to the Games article

A Failed Cartel: When Collusive Firms Become Competitors (Games 2026, 17, 56) compares outcomes across market structures. Its isoelastic coalition example finds the optimal total and verifies a convergent scalar iteration. There, identical linear costs make moving production between coalition plants irrelevant to joint profit; equal sharing specifies the split.

Here the question is *who produces how much* when plants differ and orders are already committed. The staffing plans favor distant allocations with nearly equal profits. Adjustment costs price revisions; independently varying marginal charges select a unique adjusted global plan for almost every charge pair. At each such fixed criterion, every maximizing sequence converges to that plan. This gives stable approximation of the full plant allocation despite kinks and competing peaks, without equal sharing or a contraction assumption. It supplies no algorithm or convergence rate.