

# Long-Run Equilibrium in the Market of Mobile Services in the USA

Based on Badev, Kabaivanov, Kopanov, Zhelinski and Zlatanov  
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**Main point.** The paper constructs flexible sigmoid-based empirical response functions for quarterly subscriber counts. It combines a statistical description of past growth with a mathematical test that the fitted market settles at one stable long-run allocation.

## 1. The economic object: a steady subscriber allocation

The paper uses historical U.S. data to model AT&T and Verizon as a duopoly, abstracting from smaller providers. With subscriber counts in millions, write the current state as  $w = (u, v)$  and next quarter's response as  $T(w) = (F(u, v), G(u, v))$ . A long-run state satisfies  $T(w^*) = w^*$ : neither provider's count changes under the fitted responses. These are empirical response functions, not demonstrated profit-maximizing best replies.

## 2. How the sigmoids and response functions are constructed

The share curves use  $A + BL(t - 28) + CL(t - 36)$ , where  $L(s) = (1 + e^{-s})^{-1}$ . Shared transition centers are quarters 28 and 36; the logistic rate is one. Mathematica numerical experiments select the coefficients  $(A, B, C) = (0.3275, 0.0634, 0.0557)$  for AT&T and  $(0.3411, -0.0469, 0.0017)$  for Verizon. With these dates fixed, only three coefficients per provider summarize many quarterly observations, targeting baseline levels, sizes of shifts and later levels.

A separate sigmoid targets aggregate subscriber growth and its slowdown. Shares are converted into counts; fitted count paths are inverted to recover time, advanced one quarter and combined into two-input responses  $(F_1, G_1)$ . Linearization gives  $(F_2, G_2)$ .

## 3. What contraction adds beyond fitting the data

A close fit reproduces observed history, but changed starting counts may still produce unstable futures. Run two copies of the fitted market from different starts. Within a nonempty closed region  $D$  that responses never leave, contraction requires their total absolute subscriber gap to shrink at every update:

$$\|T(w) - T(\tilde{w})\|_1 \leq \alpha \|w - \tilde{w}\|_1, \quad 0 \leq \alpha < 1.$$

Two distinct steady states cannot satisfy this: their unchanged gap would have to shrink. The fixed-point theorem guarantees existence in  $D$  and convergence from every start there, with distance to equilibrium at most  $\alpha^t$  times its initial value after  $t$  quarters. The paper checks absolute derivative column sums below one, applying a coupled fixed-point theorem for *semi-cyclic maps*. Contraction therefore makes the fitted model forget its starting point; it does not validate its forecasts.

## 4. A numerical example: the linearized response

The paper's printed linear approximation is

$$\begin{aligned} u_{t+1} &= 8.53491 + 0.895879u_t + 0.0641947v_t, \\ v_{t+1} &= 9.61236 + 0.00125026u_t + 0.895879v_t. \end{aligned}$$

From (100, 90) million subscribers, the first update is (103.900, 90.366); the limit is approximately (139.925, 93.999). For the rounded coefficients,  $D = [0, 141] \times [0, 95]$  maps into itself and the largest derivative column sum is  $\alpha = 0.9600737$ . Two states in  $D$  initially one million subscribers apart become at most 960,074 subscribers apart next quarter.

## 5. Comparison with a statistical benchmark

The benchmark directly predicts next-quarter counts using ten-term bivariate cubic regressions, fitted by least squares with derivative restrictions. Both approaches fit one-step observations reasonably well. The reported advantage of the nonlinear sigmoid model is stronger *recursive* fit, feeding its own predictions into later quarters. The paper uses Pearson and Kolmogorov–Smirnov diagnostics. Recursive evaluation windows differ, and no held-out forecast comparison is presented. The linearization illustrates stability; the nonlinear specification gives the stronger reported recursive fit. Long-run conclusions assume unchanged responses and do not establish causal competitor effects.