Set-valued best responses under generalized concavity

Equilibrium in a Failed Cartel: Set-Valued Reactions and Sources of Market Asymmetry

Anton Badev, Vanya Ivanova, Diana Nedelcheva, Martin Pavlov and Boyan Zlatanov

Main point. A positive, strictly decreasing, (-1)-concave inverse demand can give a firm an entire interval of profit-maximizing outputs. This does not imply multiple Nash equilibria. The construction below has set-valued best responses away from equilibrium, but its unique equilibrium is $(3, 3, 3)$.

1. Market primitives and generalized concavity

Three firms simultaneously choose $x_i \in [0, 3]$, $i = 1, 2, 3$. Let $Q = x_1 + x_2 + x_3$. All firms have the same cost function $c_i(x_i) = 0$ and profit $\pi_i = x_i P(Q)$. Specify inverse demand by

$$P(Q) = \begin{cases} e^{2-Q}, & 0 \le Q \le 2, \\ (Q-1)^{-1}, & Q \ge 2. \end{cases}$$

The two branches meet at $P(2) = 1$ with the same derivative, -1. Demand is positive, continuously differentiable and strictly decreasing. For positive demand, (-1)-concavity means that the reciprocal $g(Q) = 1/P(Q)$ is convex. Here

$$g(Q) = \begin{cases} e^{Q-2}, & 0 \le Q \le 2, \\ Q-1, & Q \ge 2. \end{cases}$$

Its derivative increases to one and then remains constant, establishing convexity (Figure 1).

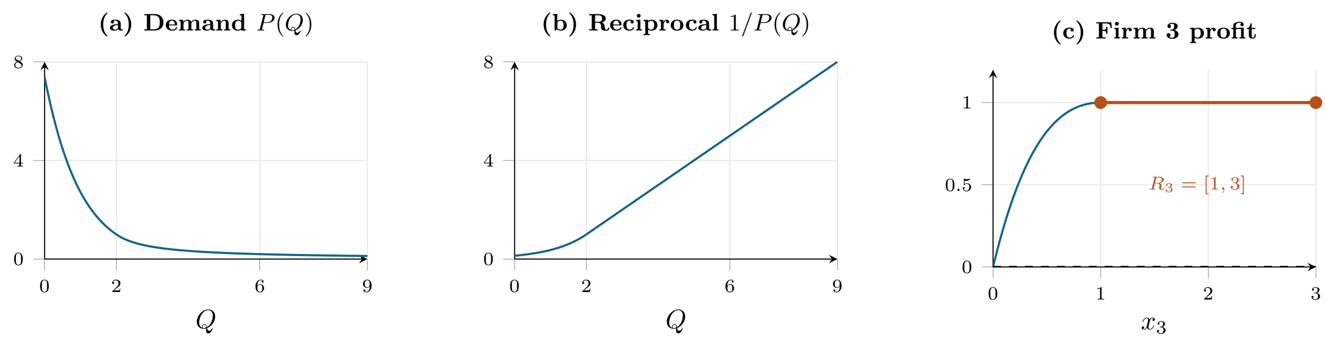

Figure 1: Demand and its convex reciprocal over feasible aggregate output $0 \le Q \le 9$. In (c), rivals produce a total of one: profit rises to one and stays there. The dashed baseline is Firm 3's zero cost.

2. An interval of optimal quantities

Fix $x_1 + x_2 = 1$, for example $x_1 = x_2 = 1/2$. Firm 3 then earns

$$\pi_3(x_3) = \begin{cases} x_3 e^{1-x_3}, & 0 \le x_3 \le 1, \\ 1, & 1 \le x_3 \le 3. \end{cases}$$

Before one, $\pi_3'(x_3) = e^{1-x_3}(1 - x_3) > 0$. Every output in $[1, 3]$ therefore maximizes profit:

$$\boxed{R_3(x_1, x_2) = [1, 3] \quad \text{whenever } x_1 + x_2 = 1.}$$

This is a quasi-concave profit function with a flat maximum, not a strictly concave objective. The same argument applies to any firm by symmetry.

Why the response plateau does not create multiple equilibria

3. Fixed rivals versus unilateral deviations

Moving along $x_1 + x_2 = 1$ changes the allocation between Firms 1 and 2 without changing Firm 3's profit curve. This line is a collection of rival profiles, **not a constraint in the Cournot game**. When Firm 1 deviates, Firm 2's output stays fixed, so their combined output changes.

At $(1/2, 1/2, 1)$, Firm 3 is best responding. Each other firm faces rival output $3/2$ and earns $\pi(x) = xe^{1/2-x}$ up to $x = 1/2$, then $\pi(x) = x/(x + 1/2)$. Both branches are strictly increasing on their relevant intervals. Either firm can raise profit from $1/2$ to $6/7$ by choosing 3. Thus this profile is not an equilibrium.

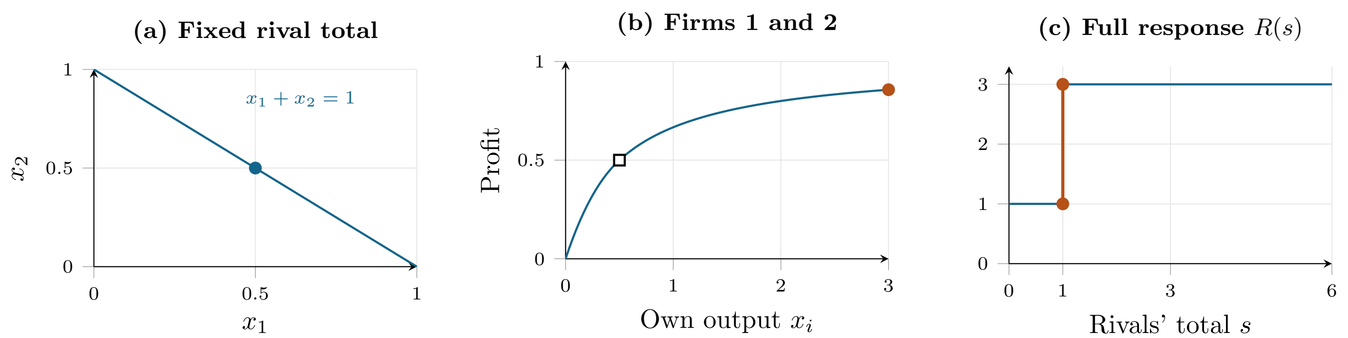

Figure 2: (a) Allocations preserving Firm 3's rival total. (b) Identical profit curves for Firms 1 and 2 at the illustrative profile: the square marks current output; the orange point marks the profitable deviation. (c) The vertical segment at $s = 1$ is the set-valued response.

4. The complete correspondence and unique equilibrium

For any firm, let s be the sum of its rivals' outputs. Where $x + s \le 2$, its profit derivative is $e^{2-s-x}(1 - x)$. Where $x + s \ge 2$, it is $(s - 1)/(x + s - 1)^2$. Comparing the branches gives

$$R(s) = \begin{cases} \{1\}, & 0 \le s < 1, \\ [1, 3], & s = 1, \\ \{3\}, & 1 < s \le 6. \end{cases}$$

Every best response is at least one. Hence, at any Nash equilibrium, every firm faces $s \ge 2$ and must choose three. Conversely, against rivals producing three each, choosing three is optimal. The unique equilibrium is therefore $(3, 3, 3)$, with $Q = 9$, $P = 1/8$ and profit $3/8$ per firm. Capacity bounds bind.

5. What the example establishes for the paper

The example illustrates the distinction between quasi-concavity, single-valued responses and uniqueness of a fixed point. Its responses have nonempty, compact, convex values and a closed graph; the jump is filled by $[1, 3]$. This fits the paper's set-valued framework (Theorem 5) and the plateau allowed by Lemma 1. Zero costs are convex and nondecreasing, but not strictly increasing: the example is not an application of any strict-cost version of Corollary 2.

It does not invoke the stronger $(-1/3)$-concavity uniqueness condition: $P^{-1/3} = (Q - 1)^{1/3}$ is strictly concave for $Q > 2$. Nor does it instantiate the anticipatory extension or Proposition 5: Firm 3 has no globally unique response, and the equilibrium is on the boundary. An anticipatory interpretation would need a specified response selection.

Manuscript reference. Badev, Ivanova, Nedelcheva, Pavlov and Zlatanov, *Equilibrium in a Failed Cartel: Set-Valued Reactions and Sources of Market Asymmetry*, supplied 43-page proofreading version, Sections 2.5–4. The present construction is a new companion example, not the manuscript's Example 2.